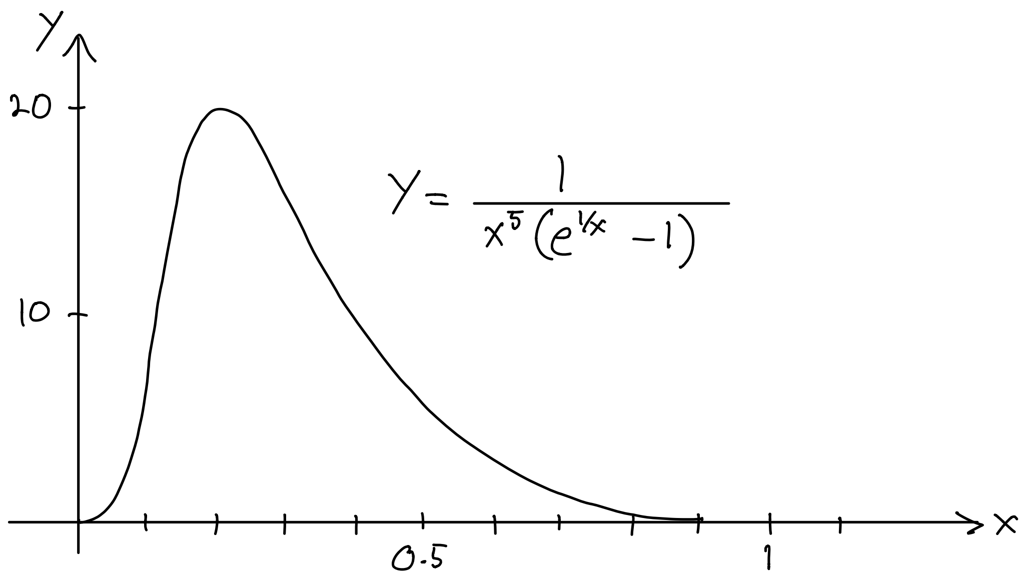
Example 6.1:

As an example we compute the maximum of the function $F: \mathbb{R}_{>0} \to \mathbb{R}$,

$$F(x) := \frac{1}{x^5(e^{1/x}-1)}$$



The function F is related to the Planck function

$$J(\lambda) = \frac{c^2 h}{\lambda^5\left(\exp\left(\frac{ch}{\lambda k T}\right)-1\right)},$$

which describes the radiation intensity of a black body at temperature T as a function of the wave length λ. (c is speed of light, h the Planck constant, and k the Boltzmann constant). Setting $x = \frac{kT}{ch}\lambda$ gives

$$\eta(\lambda) = \frac{k^5 T^5}{c^3 h^4} F(x).$$

For $x > 0$ we get

$$F'(x) = -\frac{5x^4(e^{1/x}-1) - x^3 e^{1/x}}{x^{10}(e^{1/x}-1)^2},$$

and $F'(x) = 0$ if and only if

$$5x(e^{1/x}-1) - e^{1/x} = 0$$

Substituting $t := \frac{1}{x}$, this is equivalent to

$$5(1-e^{-t}) = t.$$

$\Rightarrow$ Setting $f(t) := 5(1-e^{-t})$, we have to solve the equation $f(t) = t$. $(*)$

We first show that $(*)$ has a unique solution t^* in $\mathbb{R}_{>0}$ with $t^* \in [4,5]$.

$$f'(t) = 5e^{-t} \Rightarrow f'(t) > 1 \text{ for } t < \log 5$$

$\Rightarrow$ $f(t) - t$ is strictly monotonically increasing on $[0, \log 5]$.

As $f(0) = 0 \Rightarrow f(t) > t$ for $t \in (0, \log 5]$.

For $t > \log 5$, we have $f'(t) < 1$

$\Rightarrow$ $f(t) - t$ is strictly monotonically decreasing on $[\log 5, \infty)$

$\Rightarrow$ $f(t) = t$ at most on one point on $[\log 5, \infty)$

We compute

$$\left.\begin{array}{l} f(4) = 4.90\ldots > 4, \\ f(5) = 4.96\ldots < 5 \end{array}\right\} \Rightarrow \begin{array}{l} f(4) - 4 > 0 \\ f(5) - 5 < 0 \end{array}$$

Intermediate value th. $\Rightarrow \exists\, t^* \in [4, 5]$ with

$$f(t^*) = t^*$$

Set

$$q := \sup_{t \in [4,5]} |f'(t)| = f'(4) = 5e^{-4} = 0.09157\ldots,$$

$$\frac{q}{1-q} = 0.1008\ldots,$$

thus the sequence $t_0 := 5$, $t_{n+1} := f(t_n)$ converges against t^* with the following error estimate:

$$|t^* - t_n| \leq 0.101\, |t_n - t_{n-1}|.$$

This way we obtain after a few iterations $t^* = 4.965114\ldots$ with an error $\varepsilon = 10^{-6}$

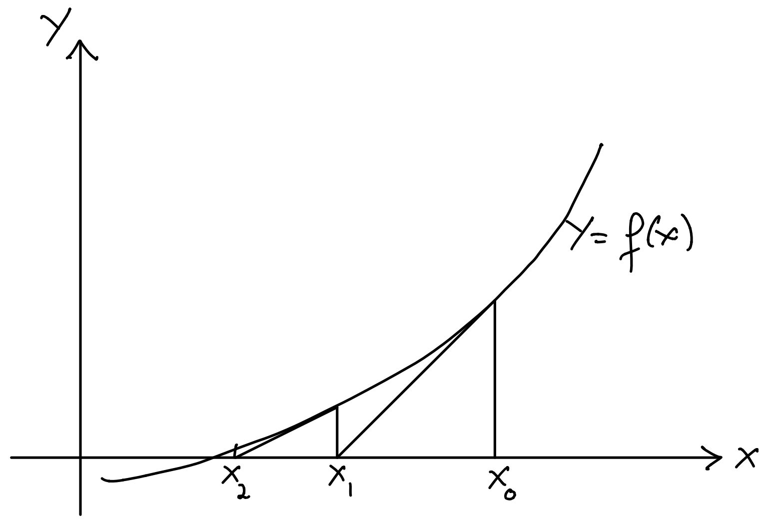
$\Rightarrow$ the solution of $F'(x) = 0$ is:

$$x^* = \frac{1}{t^*} = 0.2014052 \pm 10^{-7}$$

$$\left(\text{or } \lambda_{max} = 0.2014 \frac{ch}{KT}\right)$$

§6.2 Newton method

The Newton method for solving the equation $f(x) = 0$ uses the tangent at a value x_0 and iteratively replaces it with the intersetion with the x-axis:



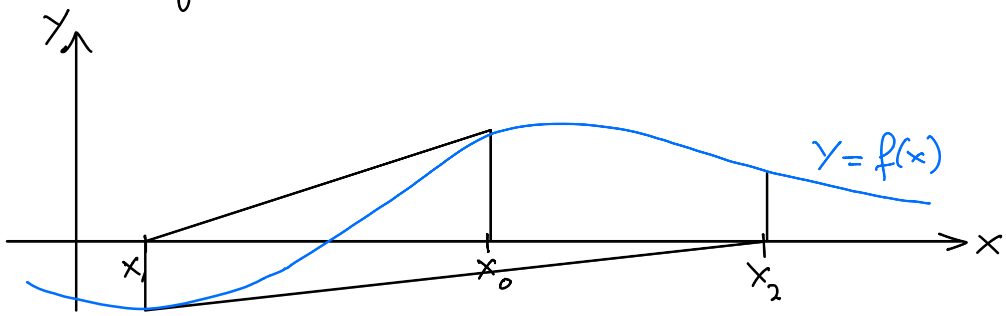
$$\Rightarrow \quad x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \in \mathbb{N}$$

$$\left(\Leftrightarrow \quad f'(x_n)\,(x_{n+1} - x_n) + f(x_n) = 0 \right)$$

Let f be defined in a closed interval D and continuously differentiable with $f'(x) \neq 0$ for all $x \in D$. For a well-defined sequence (x_n) converging against a $\xi \in D$, we get

$$\xi = \xi - \frac{f(\xi)}{f'(\xi)} \quad \Rightarrow \quad f(\xi) = 0$$

In general, this method does not need to converge:



The following Proposition describes a case, where the above sequence is convergent.

Proposition 6.2:

Let $f: [a,b] \longrightarrow \mathbb{R}$ be a twice differentiable convex function with $f(a) < 0$ and $f(b) > 0$.

Then:

i) There is exactly one $\xi \in (a,b)$ with $f(\xi) = 0$

ii) Is $x_0 \in [a,b]$ an arbitrary point with $f(x_0) \geq 0$, then the sequence

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \in \mathbb{N}$$

is well-defined, monotonically decreasing, and converges against ξ.

iii) For $f'(\xi) \geq C > 0$ and $f''(x) \leq K$ for all

$x \in (\xi, b)$ one has for $n \geq 1$ the following estimate

$$|x_{n+1} - x_n| \leq |\xi - x_n| \leq \frac{K}{2C} |x_n - x_{n-1}|^2 .$$

Remark 6.2:

i) Analogous results hold if f is conkave or if $f(a) > a$ and $f(b) < 0$.

ii) The error estimate says, that the Newton method shows "quadratic convergence". Is $\frac{K}{2C} \sim 1$, and $|x_{n-1} - x_n| < 10^{-k}$, then $|\xi - x_n| < 10^{-2k}$ and with each iteration the error halves.

Proof of Prop. 6.2:

i) As $f''(x) \geq 0$ for all $x \in (a,b)$, the function f' is monotonically increasing on $[a,b]$.
Intermediate value theorem
$\Rightarrow \exists\ q \in [a,b]$ s.t. $f(q) = \inf \{ f(x) \mid x \in [a,b] \} < 0$.
If $q \neq a$, we get $f'(q) = 0$, so $f'(x) \leq 0$ for $x \leq q$.
$\Rightarrow f$ is monotonically decreasing on $[a, q]$ and has no zero there.

$\Rightarrow$ all zeros of $f : [a,b] \to \mathbb{R}$ are on (q,b) and as $f(b) > 0$ (assumption), we get <u>at least</u> one zero of f on (q,b).

Assume there are 2 zeros: $\xi_1 < \xi_2$.

Mean value theorem

$$\Rightarrow \exists\, t \in (q, \xi_1) \text{ s.t.}$$

$$f'(t) = \frac{f(\xi_1) - f(q)}{\xi_1 - q} = \frac{-f(q)}{\xi_1 - q} > 0,$$

thus also $f'(x) > 0$ for all $x \geq \xi_1$.

$\Rightarrow$ f is strictly monotonically increasing on $[\xi_1, b]$ and thus ξ_1 is <u>only</u> zero.

ii) Let $x_0 \in [a,b]$ with $f(x_0) \geq 0$. Then $x_0 \geq \xi$.

We prove by induction that the sequence

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

satisfies $f(x_n) \geq 0$ and $\xi \leq x_n \leq x_{n-1}$ $\forall\, n \in \mathbb{N}$.

<u>$n \longrightarrow n+1$</u>:

From $x_n \geq \xi$ we get $f'(x_n) \geq f'(\xi) > 0$, also $\frac{f(x_n)}{f'(x_n)} \geq 0$ and thus $x_{n+1} \leq x_n$.

Next, we need to prove $f(x_{n+1}) \geq 0$.

To this end, look at the function

$$\varphi(x) := f(x) - f(x_n) - f'(x_n)(x - x_n)$$

Due to the monotomy of f' we have

$$\varphi'(x) = f'(x) - f'(x_n) \leq 0 \quad \text{for } x \leq x_n .$$

As $\varphi(x_n) = 0$, we get $\varphi(x) \geq 0$ for $x \leq x_n$, therefore

$$\begin{aligned} 0 \leq \varphi(x_{n+1}) &= f(x_{n+1}) - f(x_n) - f'(x_n)(x_{n+1} - x_n) \\ &= f(x_{n+1}) . \end{aligned}$$

$$\Rightarrow \quad x_{n+1} \geq \xi$$

Thus $(x_n)_{n \in \mathbb{N}}$ is monotonically decreasing and bounded from below

$$\Rightarrow \lim_{n \to \infty} x_n =: x^* \text{ exists !}$$

But $f(x^*) = 0$, thus due to uniqueness of ξ : $x^* = \xi$.

$\square$

Example 6.2:

Let $k \in \mathbb{N}$, $k \geq 2$ and $a \in \mathbb{R}_{>0}$. Consider the function

$$f: \mathbb{R}_+ \longrightarrow \mathbb{R}, \quad f(x) := x^k - a.$$

We have $f'(x) = kx^{k-1}$ and $f''(x) = k(k-1)x^{k-2} \geq 0$ for $x \geq 0$, so f convex.

$\Rightarrow$ We can apply the Newton method:

$$x - \frac{f(x)}{f'(x)} = x - \frac{x^k - a}{kx^{k-1}} = \frac{1}{k}\left((k-1)x + \frac{a}{x^{k-1}}\right)$$

Thus, for arbitrary x_0 with $x_0^k > a$ the sequence

$$x_{n+1} := \frac{1}{k}\left((k-1)x_n + \frac{a}{x_n^{k-1}}\right), \quad n \in \mathbb{N}$$

converges against $\sqrt[k]{a}$. If $x_0^k < a$, then $x_1^k > a$ and the method still converges.

Example 6.3:

Consider the function $f: \mathbb{R}_+ \longrightarrow \mathbb{R}$, $f(x) = x^3 - 2x - 5$

We observe

$$f''(x) = 6x \geq 0 \text{ for } x \geq 0$$

$\Rightarrow$ f is convex on $\mathbb{R}_+$ and $f(0) = -5 < 0$, $f(\infty) = \infty$

Thus we can apply Prop. 6.2 to find a root of $f(x) = 0$:

start with $x_0 = 2$

We have $f'(x) = 3x^2 - 2$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

$$\text{and } x_1 = x_0 - \frac{x_0^3 - 2x_0 - 5}{3x_0^2 - 2}$$

$$= 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2} = 2.1 \quad (f(2.1) = 0.061 > 0)$$

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$

$$= 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} \approx 2.0946$$

This approximation is accurate to four decimal places.

<u>Example 6.4:</u>

Consider $f : [0, \frac{\pi}{2}] \longrightarrow \mathbb{R}$, $f(x) = x - \cos x$

Then $f''(x) = \cos x \geqslant 0$ and $f(0) = -1 < 0$, $f(\frac{\pi}{2}) = \frac{\pi}{2} > 0$

$\Rightarrow$ can apply Newton's method to solve

$$f(x) = 0 \iff \cos x = x$$

We compute $f'(x) = 1 + \sin x$, so

$$x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n}$$

For $x_0 = 1$ $(f(x_0) > 0)$, we get

$x_1 \sim 0.75036387$

$x_2 \sim 0.73911289$

$x_3 \sim 0.73908513$

$x_4 \sim 0.73908513$

$\rightarrow$ very accurate after only a few steps!